Linear Algebra

[KOMS120301] - 2023/2024

14.2 - Eigenvector

Dewi Sintiari

Computer Science Study Program
Universitas Pendidikan Ganesha

Week 14 (December 2023)

Learning objectives

After this lecture, you should be able to:

1. explain the concept of eigenvalues and eigenvectors;
2. find the eigenvalues of a matrix;
3. find the eigenvectors of a matrix;
4. find the bases of eigenspace of a matrix.

Motivating example

Part 1: Eigenvectors & Eigenvalues

What are eigenvectors & eigenvalues?

Definition

Let A be an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an eigenvector of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is:

$$A\mathbf{x} = \lambda\mathbf{x}$$

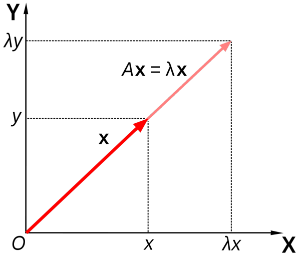
for some scalar $\lambda \in \mathbb{R}$.

λ is called an eigenvalue of A (or of T_A), and $\mathbf{x}$ is said eigenvector corresponding to λ.

Geometric interpretation

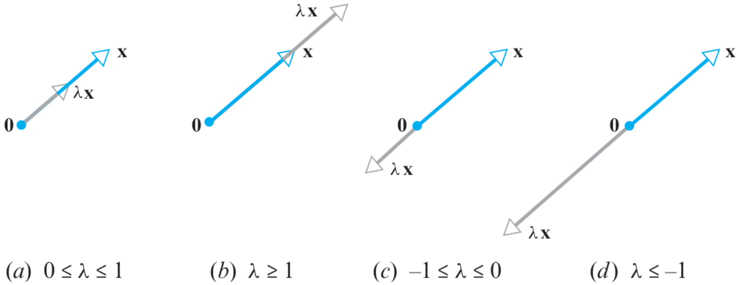
The eigenvector **x** represents:

the column vector in which multiplying it by a square matrix A yields a vector $\lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$, i.e.
a vector that is a multiplication of **x**
(same direction as **x** but with different magnitude).



Geometric interpretation

Depending on the sign and magnitude of the eigenvalue λ corresponding to $\mathbf{x}$, the operation $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches $\mathbf{x}$ by a factor of λ.



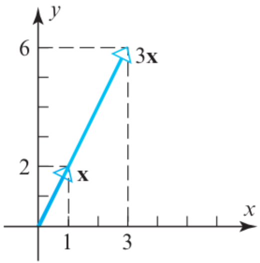
Example

Given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$.

Example

Given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$.

$$A\mathbf{x} = A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$



Exercise

Part 2: Computing Eigenvalue

How to compute eigenvalues?

Example

How to get the value $\lambda = 3$ and the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ from the previous example?

Recall that an eigenvalue λ and an eigenvector $\mathbf{x}$ of A must satisfy

$$A\mathbf{x} = \lambda\mathbf{x}$$

Hence,

$$A\mathbf{x} = \lambda\mathbf{x} \;\Leftrightarrow\; AI\mathbf{x} = \lambda I\mathbf{x} \;\Leftrightarrow\; A\mathbf{x} = \lambda I\mathbf{x} \;\Leftrightarrow\; (\lambda I - A)\mathbf{x} = 0$$

Recall that $(\lambda I - A)\mathbf{x} = 0$ has a non-zero solution when

$$\det(\lambda I - A) = 0$$

How to compute eigenvalues?

Theorem (Eigenvalue)

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation:

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A.

Example: how to get the eigenvalue?

Given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. By the theorem, we solve $\det(\lambda I - A) = 0$, that is:

Example: how to get the eigenvalue?

Given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. By the theorem, we solve $\det(\lambda I - A) = 0$, that is:

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0 \iff \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

which yields:

$$(\lambda - 3)(\lambda + 1) = 0 \iff \lambda_1 = 3 \text{ and } \lambda_2 = -1$$

This means that the eigenvalues of A are 3 and -1.

Generalization

For a matrix A of size $n \times n$, the charateristic equation $(\lambda I - A)\mathbf{x} = 0$ yields:

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0 \tag{1}$$

The polynomial: $(\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n)$ is called the characteristic polynomial of A.

Example

The characteristic polynomial of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ is

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

Exercise 1: Eigenvalues of a 3×3 matrix

Find the eigenvalues of:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution:

Exercise 1: Eigenvalues of a 3×3 matrix

Find the eigenvalues of:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution:
Compute the characteristic polynomial:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The eigenvalues are the solution of:

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

that is:

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0 \iff \lambda_1 = 4, \ \lambda_2 = 2 + \sqrt{3}, \text{ and } \lambda_3 = 2 - \sqrt{3}$$

Exercise 2: Eigenvalues of an **upper** triangular matrix

Given: $A = \begin{bmatrix} \frac{1}{2} & -1 & 5 \\ 0 & \frac{2}{3} & -8 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$. Find the eigenvalues of A.

Exercise 3: Eigenvalues of a **lower** triangular matrix

Given: $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$. Find the eigenvalues of A.

What can you say about the eigenvalues of a **triangular matrix**?

Find the eigenvalues of:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

What can you say about the eigenvalues of a **triangular matrix**?

Find the eigenvalues of:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Solution:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix} \\ &= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) \end{aligned}$$

Hence the characteristic equation is:

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

that gives $\lambda_1 = a_{11}, \; \lambda_2 = a_{22}, \; \lambda_3 = a_{33}, \; \lambda_4 = a_{44}$

Does it hold for **diagonal matrices**?

Yes, because a diagonal matrix is a triangular matrix.

Part 3: Computing Eigenvectors

Recap

So far, we have seen...

Theorem

If A is an $n \times n$ matrix, the following statements are equivalent.

1. *λ is an eigenvalue of A.*
2. *λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.*
3. *The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.*
4. *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

We have seen 1, 2, and 3. Now we will see that 4 holds.

Finding eigenvectors (1)

By definition, the eigenvectors of A corresponding to an eigenvalue λ are the **nonzero** vectors that satisfy:

$$(\lambda I - A)\mathbf{x} = 0$$

Example

In the previous example, we are given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ with eigenvalues 3 and -1.

We can compute the eigenvector for each eigenvalue by solving:

1. $(3I - A)\mathbf{x} = \mathbf{0}$;
2. $(-I - A)\mathbf{x} = \mathbf{0}$;

Finding eigenvectors (2)

For $\lambda = 3$

$$
\begin{aligned}
(3I - A)\mathbf{x} &= 0 \\
\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -8x_1 + 4x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Hence, it must be that $-8x_1 + 4x_2 = 0 \iff x_1 = \frac{1}{2}x_2$. The parametric solution is $x_1 = s,\ x_2 = 2s$ with $s \in \mathbb{R} \setminus \{0\}$.

Finding eigenvectors (3)

For $\lambda = -1$

$$(-I - A)\mathbf{x} = 0$$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4x_1 \\ -8x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $x_1 = 0$ and $x_2 = t$ with $t \in \mathbb{R} \setminus \{0\}$.

So, can you explain the step-by-step computing the eigenvalues and the eigenvectors?

To compute the **eigenvalues**, we...

To compute the **eigenvectors**, we...

Part 4: Bases for eigenspaces

What is eigenspace?

Note that the eigenvector of A corresponding to λ is the solution of the **linear system**:

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

So an eigenvector $\mathbf{x}$ is a nonzero vector in the solution space of the linear system.

The solution space of the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the eigenspace of A.

The eigenspace of A corresponding to λ can be viewed as:

1. the null space of the matrix $\lambda I - A$;
2. the kernel of the matrix operator $T_{(\lambda I - A)} : \mathbb{R}^n \to \mathbb{R}^n$;
3. the set of vectors for which $A\mathbf{x} = \lambda\mathbf{x}$

Example: how to find an eigenspace?

Look again at the previous example.

We are given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ with eigenvalues 3 and -1.

- For $\lambda = 3$, the eigenvectors are determined by:

$$x_1 = s,\ x_2 = 2s \text{ with } s \in \mathbb{R} \setminus \{0\} \quad \text{or} \quad \mathbf{x}_1 = \begin{bmatrix} s \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- For $\lambda = -1$, the eigenvectors are determined by:

$$x_1 = 0 \text{ and } x_2 = t \in \mathbb{R} \setminus \{0\} \quad \text{or} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -1$.

Exercises

Exercise 1.

Find bases for eigenspaces of the matrix:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Exercise 2.

Find bases for eigenspaces of the matrix:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Part 5: Eigenvalues and invertibility

Motivating example

Question 1.

We have seen (in the previous example) that the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ has eigenvalues 3 and -1.

Task: Compute $\det(A)$.

Question 2.

Given matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Task:

- Compute the eigenvalues of A.
- Compute the determinant of A

So, what can you say about the relation between the determinant of A and the eigenvalues of A?

to be continued...